Constant-time Lattice Reduction for SQIsign

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SQIsign: A post-quantum signature

- Post-quantum signature

  De Feo, Kohel, Leroux, Petit and Wesolowski, 2020

Eprint 2020/1240
Logo from sqisign.org
LVL1 data from pqsort.tii.ae (24/02/2025)

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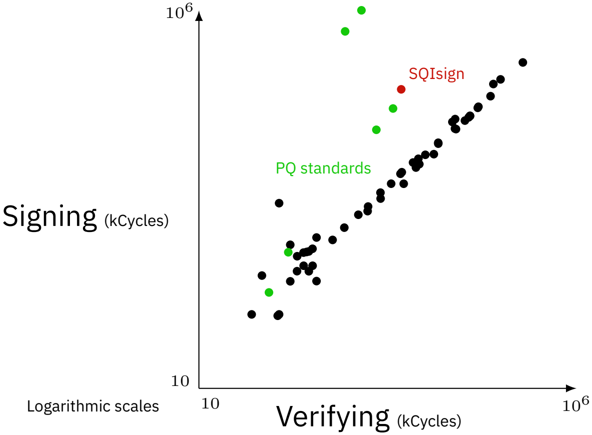

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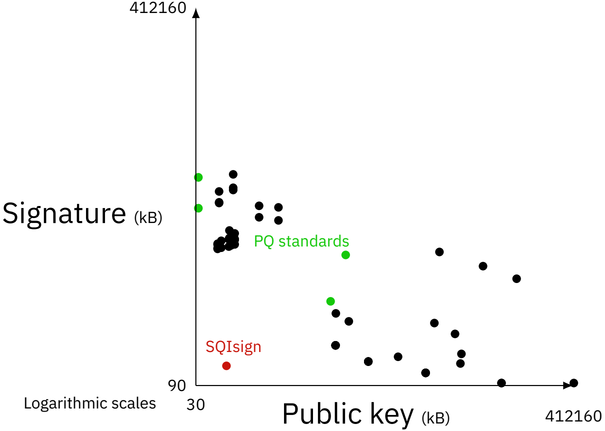
+ signatures: 65 bytes



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Short Quaternion and Isogeny Signature

Quaternions and Isogenies

	Secret World Quaternions	**Public World** Isogenies
Objects	Ideals	Isogenies
Lengths	Ideal norm	Isogeny degree

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short ideal $\rightarrow$ short isogeny

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short ideal $\rightarrow$ short isogeny

short element in ideal $\rightarrow$ short ideal

Quaternion ideals

Quaternion algebra:

- 4-dimensional vector space over $\mathbb{Q}$
+ ...
+ with quadratic form called norm N

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Ideal:

- rank 4 lattice
  - Lattice: set of $\mathbb{Z}$-linear combinations of a $\mathbb{Q}$-basis

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- \+ ...
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Ideal:

- rank 4 lattice
  - Lattice: set of $\mathbb{Z}$-linear combinations of a $\mathbb{Q}$-basis
- \+ ...
- \+ integer norm: $N(I) = \gcd_{x \in I} N(x)$

The lattice reduction problem

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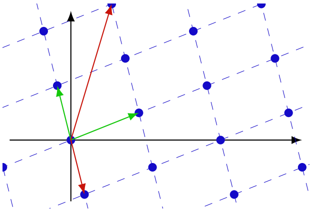
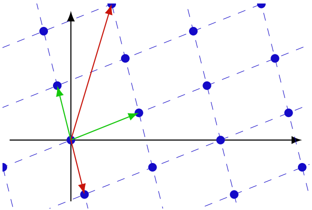


The lattice reduction problem

Lattice: Set of $\mathbb{Z}$-linear combinations of a $\mathbb{Q}$-basis of $\mathbb{Q}^4$

Given: A lattice basis B

Find: A *reduced* basis B' of the same lattice as B



Reduced: Containing only vectors which are

- of somewhat small norm
- and somewhat orthogonal

Several definitions exist

Lattice reduction like in lattice cryptography?

Lattice-crypto:

- Large dimension over $\mathbb{Q}$
- Often smaller integers

$\rightarrow$ Optimization for high dimension

- Lattice reduction in cryptanalysis

$\rightarrow$ Need fast reduction

SQIsign:

- Dimension 4 over $\mathbb{Q}$
- Large integers

$\rightarrow$ Optimization for large coefficients

- Lattice reduction used constructively

$\rightarrow$ Need secure reduction

Practical security in cryptography

Mathematical security:
Ensure that given only public information, an adversary cannot break the system

Implementation security:
Ensure observing the computation only gives public information to an adversary

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- Runtime
- Power consumption
- Memory accesses
- Faults

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Constant-time algorithms

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Assume basic arithmetic is constant-time

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good	Naive matrix multiplication
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Constant-time algorithms

Constant-time: Secret-independent runtime

Assume basic arithmetic is constant-time

good Naive matrix multiplication
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Simplified: No if, no while

Lattice reduction algorithm: LLL

Require: B basis of a lattice L of rank d and $c \in]1/4, 1[$
Ensure: B an c-LLL-reduced basis of L

$B^* :=$ Gram-Schmidt-Orthogonalize(B)
while B is not reduced **do**
    Size-reduce B , update B^*
    for i from 1 to $d-1$ **do**
        if not LLLcondition(c, i, B, B^*) **then**
            Swap b_i, b_{i+1} in B, update B^*, continue
        end if
    end for
end while
return B

From “Factoring polynomials with rational coefficients” by A. Lenstra, H. Lenstra, L. Lovász, 1982
Description adapted from H. Cohen’s “A Course in Computational Algebraic Number Theory”, 1993

Lattice reduction algorithm: Greedy

Require: B basis of a lattice L of rank $d \leq 4$, G its Gram matrix
Ensure: B a Minkowski-reduced basis of L, G its Gram matrix

1: done $:=$ False
2: **while** not done and $d > 1$ **do**
3: Sort $(b_1, ..., b_d)$ by norm, adapt B and G;
4: $b_1, ..., b_{d-1}, G' :=$ Greedy$(b_1, ..., b_{d-1})$ adapt B and G
5: $b_d := b_d - c$ where c is closest to b_d in the lattice of $b_1, ..., b_{d-1}$, adapt B and G;
6: done $:= (N(b_d) \geq N(b_{d-1}))$
7: **end while**
8: **return** B, G

From "Low-Dimensional Lattice Basis Reduction Revisited" by P. Q. Nguyen, D. Stehlé, 2004
(DOI 10.1007/978-3-540-2)

Lattice reduction algorithm: BKZ-2

Require: B basis of a lattice L of rank 4, parameter $\delta < 1$
Ensure: B a reduced basis of L

LLL-reduce(B)
while First tour or B has changed in previous tour **do**
    for i from 1 to 3 **do**
        b :=SVP(b_i, b_{i+1})
        if δ-condition(b, B) **then**
            Insert b in B
        end if
        LLL-reduce(B)
    end for
end while
return B

From "Lattice basis reduction: Improved practical algorithms and solving subset sum problems."
by C. P. Schnorr and M. Euchner, 1991 (DOI 10.1007/3-540-54458-5_51); Description from Eprint 2011/198

Lattice reduction algorithm: BKZ-2 for analysis

Require: B basis of a lattice L of rank 4, optional T_m max iteration number
Ensure: B a reduced basis of L if T_m large enough

while B has changed in previous tour and T_m not reached **do**
    for i from 1 to 3 **do**
        $b_1, b_{i+1} :=$ HKZ-reduce(b_i, b_{i+1})
        Size-reduce(B)
    end for
end while
return B

From "Terminating BKZ" by G. Hanrot, X. Pujol and D. Stehlé, 2011
(Eprint 2011/198)

Constant-time BKZ-2

Require: B basis of a lattice L of rank 4, iteration counts T_{Lagr}, T_{BKZ}
Ensure: B' a reduced basis of L if T_{Lagr}, T_{BKZ} are large enough

```
B', G, B := B, its Gram matrix, its orthogonalization
for j from 1 to T_BKZ do
    for i from 1 to 3 do
        Size-reduce b'_i, adapt G and B*;
        Constant-time Lagrange-reduce (b'_i, b'_{i+1}, T_Lagr), adapt B', B*, G
        Size-reduce b'_i then b'_{i+1} , adapt G and B*
    end for
end for
return B'
```

Lagrange reduction

Require: b_1, b_2 basis of a rank-2 lattice L
Ensure: c, d basis of L with c minimal in L

```
while first round or N(d) < N(c) do
    μ = ⌊(N(c+d) − N(c) − N(d)) / (2N(d))⌉
    c, d := d, c − μd
end while
return c, d
```

$\quad$ **while** first round or $N(d) < N(c)$ **do**
$\quad\quad \mu = \left\lfloor \frac{N(c+d)-N(c)-N(d)}{2N(d)} \right\rceil$
$\quad\quad c, d := d, c - \mu d$
$\quad$ **end while**
$\quad$ **return** c, d

N square of a norm (quadratic form)

Lagrange reduction

Require: b_1, b_2 basis of a rank-2 lattice L
Ensure: c, d basis of L with c minimal in L

1: **for** $1, \ldots, T_{\text{Lagr}}$ **do**
2: $\quad \mu = \left\lfloor \frac{N(c+d) - N(c) - N(d)}{2N(d)} \right\rceil$
3: $\quad c, d := d, c - \mu d$
4: **end for**
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3: $\quad c, d := d, c - \mu d$
4: **end for**
5: CT-CONDITIONAL-SWAP$_{N(d)<N(c)}(c, d)$
6: **return** c, d

N square of a norm (quadratic form)

Iteration counts

Ensure:

$\|b_0\| \leqslant 2\left(\frac{4}{3}\right)^{3/2} D^{1/4}$

Require:

$$T_{\mathsf{BKZ}} \geq \frac{2}{\log_2(8/7)} \log_2\left(\log_2\left(\frac{B^*}{D^{1/4}}\right) + \sqrt{5}(\log(4/3))^{1/2}\right)$$

$$T_{\mathsf{Lagr}} \geq 2 + 2\lceil(\log_{\sqrt{3}} 2)\,(9 \log_2 B + 12)\rceil$$

B: Square root of largest diagonal coefficient of Gram matrix of the input
B^*: Square root of largest norm of a vector in the orthogonalization of the input
D: Lattice volume

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$T_{\mathsf{Lagr}} \geq 2 + 2\lceil (\log_{\sqrt{3}} 2)\,(9\log_2 B + 12)\rceil$

In SQIsign:
$T_{\mathsf{BKZ}} \approx$ Tens to hundreds

$T_{\mathsf{Lagr}} \approx$ Thousands

B: Square root of largest diagonal coefficient of Gram matrix of the input
B^*: Square root of largest norm of a vector in the orthogonalization of the input
D: Lattice volume

Gap to practice

Example:

bitsize $N(I)$	T_{Lagr} theory	T_{Lagr} no failure observed	T_{BKZ} theory	T_{BKZ} no failure observed
260	2986	9	64	3

Possible reasons:

? LLL analysis not tight

? Worst case never met

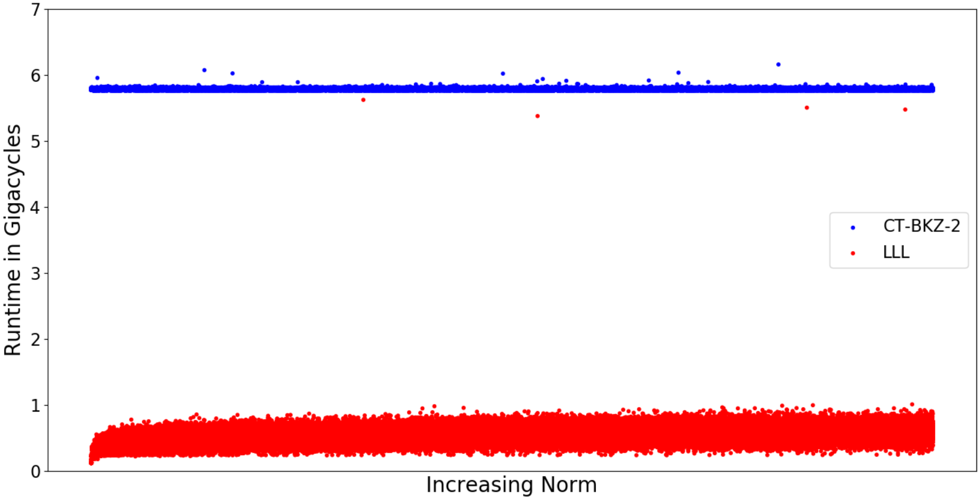
And if we did less iterations?

+ Guaranteed constant runtime
+ Output is lattice basis
– Output might not be sufficiently reduced

→ Running with reduced tours where risk is acceptable
→ Reasonable runtime

→ Experiments possible

CT-BKZ-2 is constant-time



Benchmarks and comparisons

Algorithm	old LLL	CT-BKZ-2		
Parameters			T_{BKZ}	T_{Lagr}
LVL1	7,39	35,9	4	10
LVL3	17,2	81,8	4	11
LVL5	26,4	133	4	13

All timings in gigacycles on an Intel i7-11850H processor

Benchmarks and comparisons

Algorithm	old LLL	CT-BKZ-2
Integers	CT	CT
LVL1	7,39 37 words	35,9 37 words
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LVL5	26,4 72 words	133 72 words

All timings in gigacycles on an Intel i7-11850H processor

Iterations for less than 1 failure on 100 random SQIsign-typical inputs. Such inputs are bases in HNF of random O_0-ideals of suitable norm.
For typical SQIsign inputs (from NIST round 1 or 2 implementation): $\log_2(N(I))$ about $260, 385, 508$ and $\log_2(p)$ equal $254, 378, 502$ respectively.

Benchmarks and comparisons

Algorithm	old LLL	CT-BKZ-2	
Integers	CT	CT	short CT
LVL1	7,39 37 words	35,9 37 words	9,59 20 words
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LVL1	7,39 37 words	0,0016	35,9 37 words	9,59 20 words
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LVL5	26,4 72 words	0,0031	133 72 words	38,9 37 words

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Integers and other limitations

- Current bottleneck: constant-time GCD
  - Rationals
  - Constant-time GCD self-implemented
  - Very large numbers
- Other number types?

Possible improvements

Optimizations

$\longrightarrow$ Use compiler optimization

$\longrightarrow$ Other number types

Usability

$\longrightarrow$ Failure rate estimation

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Questions?

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